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Explicit analytic solution for similarity boundary layer equations

Shi-Jun Liao [a,*], Ioan Pop [b]

[a] School of Naval Architecture and Ocean Engineering, Shanghai Jiao Tong University, Shanghai 200030, China

[b] Faculty of Mathematics, University of Cluj, R-3400 Cluj, CP253, Romania

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Abstract

In this paper the homotopy analysis method for strongly non-linear problems is employed to give two kinds of explicit analytic solutions of similarity boundary-layer equations. The analytic solutions are explicitly expressed by recurrence formulas for constant coefficients and can give accurate results in the whole regions of physical parameters.

Keywords: Analytic solution; Boundary-layer; Porous medium; Stretching wall; Homotopy analysis method

1. Introduction

Recent interest in the study of convective flow in fluid-saturated porous media has been mainly motivated by its importance in many natural and industrial problems. Numerous authors cite a wide variety of applications involving convective transport in porous media that include utilization of geothermal energy, oil reservoir modelling, building insulation, food processing and grain storage, fiber and granular insulations, contaminant transport in groundwater, casting and welding in manufacturing processes, nuclear engineering, dispersion of chemical contaminants in various industrial processes and in the environment, design of packed bed reactors and underground disposal of nuclear waste materials, etc. Several others investigate the intricate nature of solution structure from a fundamental point of view in idealized settings. This topic is therefore of vital importance to these applications, thereby generating the need for their fully understanding. Much of the recent work on this topic is reviewed by Ingham and Pop [1,2], Nield and Bejan [3], Vafai [4], and Pop and Ingham [5]. Further, the study of laminar boundary layer flow of an incompressible fluid due to a stretching surface has several important engineering applications such as the aerodynamic extrusion of plastic sheets, the cooling of an infinite metallic plate in a cooling bath, the boundary layer along liquid film condensation process, glass and polymer industries. Crane [6] produced the first study regarding the boundary layer behavior on a plane surface stretching in a viscous and incompressible quiescent fluid. The work of Crane [6] was extended by Banks [7] and more recently by Magyari and Keller [8]. The above commentary both motivates and informs the work that follows which seeks to extend limited range of understanding for free convection boundary-layer flows over a vertical flat plate embedded in a fluid-saturated porous medium or above a stretching wall in a viscous and incompressible fluid, governed by a non-linear differential equation. The aim is to explore the existence of fundamental similarity solutions based on the homotopy analysis method [9–14] proposed by Liao [9] to some viscous flow problems. In view of this success in establishing an analytic solution of a series of problems, the analytic solution for the present problems provide detailed information on the flow and heat transfer

* Corresponding author. Tel.: +86-21-6293-2676; fax: +86-21-6293-3156.

E-mail addresses: sjliao@sjtu.edu.cn (S.-J. Liao), popi@math.ubbcluj.ro (I. Pop).

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Nomenclature

a, b, A	constants
f, F	reduced stream functions
T	fluid temperature
u, v	velocity components along x and y directions, respectively
$w(x)$	velocity of the stretching wall
x, y	Cartesian coordinates along the plate and normal to it, respectively
Greek symbols	
β	parameter defined by (16)
κ	power law index
θ	reduced fluid temperature
ξ, η	similarity variables
ν	kinematic viscosity
τ	non-dimensional fluid temperature
ψ	non-dimensional stream function
Subscripts	
w	wall condition
∞	ambient temperature condition
Superscript	
$'$	differentiation with respect to similarity variable

characteristics in the whole ranges of physical parameters. The present paper intends to offer a comprehensive account of the boundary layers over a vertical flat plate embedded in a porous medium or due to a stretching wall because of their omni-presence in engineering applications.

2. Basic boundary-layer equations

Consider the steady free convection flow over a vertical semi-infinite flat plate, which is embedded in a fluid-saturated porous medium of ambient temperature T_∞. It is assumed that the temperature of the plate varies as $T_w = T_\infty + A\bar{x}^\kappa$, where $\bar{x}$ is the dimensional distance measured along the plate and A and κ are prescribed constants. It is also assumed that the porous medium is homogeneous and isotropic, that all properties of the fluid and porous medium are constant, that the fluid velocity obeys Darcy's law and that the Boussinesq approximation is valid. Let (x, y) be the non-dimensional Cartesian coordinates measured along the plate and normal to it, respectively, with the origin at the leading edge. We define the non-dimensional stream function ψ defined as

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x},$$

where (u, v) are the non-dimensional velocity components along x and y axis. We also define the non-dimensional temperature τ of the form

$$\tau = \frac{T - T_\infty}{T_w - T_\infty}, \tag{1}$$

where T is the fluid temperature and T_w, T_∞ are the temperatures on wall and at infinity, respectively. Let $\tau_w = x^\kappa$ denote the non-dimensional wall temperature. Then, assuming high Rayleigh numbers and that the boundary layer approximation holds, the problem under consideration is governed by the following equations, see Nield and Bejan [3] or Pop and Ingham [5],

$$\frac{\partial \psi}{\partial y} = \tau, \tag{2}$$

$$\frac{\partial \psi}{\partial y}\frac{\partial \tau}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \tau}{\partial y} = \frac{\partial^2 \tau}{\partial y^2}, \tag{3}$$

subject to the boundary conditions

$$\psi = 0, \quad \tau = x^\kappa, \quad \text{on } y = 0, \tag{4}$$

$$\tau = 0, \quad \text{as } y \to +\infty. \tag{5}$$

Under the transformation

$$\psi = x^{\frac{\kappa+1}{2}} f(\eta), \quad \tau = x^\kappa \theta(\eta), \quad \eta = y x^{\frac{\kappa-1}{2}}, \tag{6}$$

Eqs. (2) and (3) become

$$f' = \theta, \tag{7}$$

$$\theta'' + \left(\frac{\kappa+1}{2}\right) f\theta' - \kappa f'\theta = 0, \tag{8}$$

and the boundary conditions (4) and (5) become

$$f(0) = 0, \quad \theta(0) = 1, \quad \theta(+\infty) = 0, \tag{9}$$

where the prime denotes the differentiation with respect to η. The above equations can be combined as the so-called Cheng and Minkowycz's [15] equation

$$f'''(\eta) + \left(\frac{\kappa+1}{2}\right) f(\eta) f''(\eta) - \kappa f'^2(\eta) = 0 \tag{10}$$

subject to

$$f(0) = 0, \quad f'(0) = 1, \quad f'(+\infty) = 0. \tag{11}$$

For details, please refer to Ingham and Brown [16]. Ingham and Brown [16] proved that it holds

$2\kappa + 1 > 0,$

i.e.

$$-\frac{1}{2} < \kappa < +\infty. \tag{12}$$

Under the transformation

$$f(\eta) = \left(\frac{2}{\kappa+1}\right)^{1/2} F(\xi), \quad \xi = \eta\left(\frac{\kappa+1}{2}\right)^{1/2}, \tag{13}$$

Eq. (10) becomes

$$F'''(\xi) + F(\xi)F''(\xi) - \beta F'^2(\xi) = 0, \tag{14}$$

and the boundary conditions (11) become

$$F(0) = 0, \quad F'(0) = 1, \quad F'(+\infty) = 0, \tag{15}$$

where the prime denotes the differentiation with respect to ξ and

$$\beta = \frac{2\kappa}{\kappa+1}. \tag{16}$$

Due to (12), it holds

$$-2 < \beta \leqslant 2 \tag{17}$$

for the boundary layer flows in a porous medium.

It is interesting that the same equations as (14) and (15) were derived to describe the boundary layer flows over a stretching wall [6–8,17,18]. We consider now the two-dimensional boundary layer flow in the region $y > 0$, where (x, y) denotes a Cartesian coordinate system and the flow results solely from the movement of an impermeable flat plate at $y = 0$ in its plane. It is assumed that the speed of the boundary layer is given by $w(x)$ and the flow is such that the boundary layer equations are appropriate. Under these assumptions, the basic equations are, see Banks [7]

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{18}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{19}$$

subject to the boundary conditions

$$u = w(x), \quad v = 0 \quad \text{at } y = 0, \tag{20}$$

$$u \to 0 \quad \text{as } y \to +\infty, \tag{21}$$

where ν is the kinematic viscosity and u, v are the velocity components in the directions of increasing x, y, respectively. Assume the stretching velocity of the plate is

$$w(x) = a(x+b)^{\kappa}, \tag{22}$$

where a and b are constants. Then, under the transformation

$$\psi = \sqrt{\frac{2\nu}{a(\kappa+1)}}a(x+b)^{\frac{1}{2(\kappa+1)}}F(\xi),$$
$$\xi = \sqrt{\frac{a(\kappa+1)}{2\nu}}y(x+b)^{\frac{1}{2(\kappa-1)}}, \tag{23}$$

where ψ is the stream function defined in the usual way as mention before, Eqs. (18) and (19) become

$$F'''(\xi) + F(\xi)F''(\xi) - \beta F'^2(\xi) = 0, \tag{24}$$

and the boundary conditions (20) and (21) become

$$F(0) = 0, \quad F'(0) = 1, \quad F'(+\infty) = 0, \tag{25}$$

which are exactly the same as (14) and (15), respectively, with the same definition (16) for β. Note that $F(\xi)$ has now physical meanings when $\kappa < -1$, i.e. $2 < \beta < +\infty$. Banks [7] showed that there is no solution when $\kappa = -1/2$, corresponding to $\beta = -2$. Ingham and Brown [16] rigorously proved this point. So, the difference is only the region of β: $-2 < \beta \leqslant 2$ for flows in a porous medium (corresponding to $-1/2 < \kappa < +\infty$) but $-2 < \beta < +\infty$ for a stretching wall (corresponding to $-1/2 < \kappa < +\infty$ and $-\infty < \kappa < -1$), respectively. Banks [7] gave numerical results of above equations for $-2 \leqslant \beta \leqslant 202$ with the property $F'(\xi) \geqslant 0$, which is exactly the same as the first branch of numerical solutions given by Ingham and Brown [16]. However, unlike Ingham and Brown [16], Banks [7] did not find any dual solutions.

Some exact solutions of (14) and (15) were reported by Crane [6]. When $\beta = 1$, corresponding to $\kappa = 1$, one has the exact solution

$$F(\xi) = 1 - \exp(-\xi), \quad F''(0) = 1. \tag{26}$$

When $\beta = -1$, corresponding to $\kappa = -1/3$, one has the exact solution

$$F(\xi) = \sqrt{2}\tanh(\xi/\sqrt{2}), \quad F''(0) = 0. \tag{27}$$

So far, to the best of the authors' knowledge, the explicit analytic solutions of Eqs. (14) and (15), uniformly valid for $0 \leqslant \xi < +\infty$ and all possible values of the physical parameter $-2 < \beta < +\infty$, have not been reported. In this paper the homotopy analysis method [9–11,13,14] is applied to derive two kinds of explicit analytic solutions of Eqs. (14) and (15). Unlike perturbation techniques, the homotopy analysis method does not depend upon small parameters at all. Besides, unlike *all* previous analytic techniques, the homotopy analysis method provides us with a simple way to *control* convergence of approximation series and to *adjust* convergence regions when necessary. In this paper the homotopy analysis method is applied to solve the viscous flows of free convection in a porous medium or boundary layer flows above a stretching wall, governed by the non-linear equation (14) with the boundary conditions (15).

3. Explicit analytic solution

3.1. Free convection over a vertical flat plate embedded in a porous medium

Under the transformations

$$\zeta = \lambda\xi, \quad s(\zeta) = F(\xi), \tag{28}$$

the original equations (14) and (15) become

$$\lambda s'''(\zeta) + s(\zeta)s''(\zeta) - \beta s'^2(\zeta) = 0, \tag{29}$$

with the boundary conditions

$$s(0) = 0, \quad \lambda s'(0) = 1, \quad s'(+\infty) = 0, \tag{30}$$

where the prime denotes the differentiation with respect to ζ.

Due to the boundary conditions (30), $s(\zeta)$ can be expressed by a set of base functions

$$\{\exp(-n\zeta) | n \geqslant 0\} \tag{31}$$

in the following form

$$s(\zeta) = c_0 + \sum_{n=1}^{+\infty} c_n \exp(-n\zeta), \tag{32}$$

where c_n is coefficient. This provides us with the *Rule of Solution Expression A*, which plays an important role in the frame of the homotopy analysis method, as shown by Liao [13].

Due to the boundary conditions (30) and under the *Rule of Solution Expression A* described by (32), it is straightforward to choose

$$s_0(\xi) = [1 - \exp(-\zeta)]/\lambda_0 \tag{33}$$

as the initial guess of $s(\zeta)$, where λ_0 is the initial approximation of the coefficient λ. Besides, due to *Rule of Solution Expression A* and the governing equation (29), we choose

$$\mathscr{L}[\Phi(\zeta;q)] = \lambda_0\left[\frac{\partial^3\Phi(\zeta;q)}{\partial\zeta^3} - \frac{\partial\Phi(\zeta;q)}{\partial\zeta}\right], \tag{34}$$

as our auxiliary linear operator, where q is an embedding parameter. Note that the auxiliary linear operator $\mathscr{L}$ has the property

$$\mathscr{L}[C_1 + C_2\exp(-\zeta) + C_3\exp(\zeta)] = 0. \tag{35}$$

Furthermore, due to (29), we define such a non-linear operator

$$\mathscr{N}[\Phi(\zeta;q), \Lambda(q)] = \Lambda(q)\frac{\partial^3\Phi(\zeta;q)}{\partial\zeta^3} + \Phi(\zeta;q)\frac{\partial^2\Phi(\zeta;q)}{\partial\zeta^2} - \beta\left[\frac{\partial\Phi(\zeta;q)}{\partial\zeta}\right]^2. \tag{36}$$

Then, introducing a non-zero auxiliary parameter $\hbar$, we construct the zero-order deformation equations

$$(1-q)\mathscr{L}[\Phi(\zeta;q) - s_0(\zeta)] = \hbar q \mathscr{N}[\Phi(\zeta;q), \Lambda(q)], \quad \zeta \geqslant 0, \ q \in [0,1], \tag{37}$$

subject to the boundary conditions

$$\Phi(0;q) = 0, \quad \left.\frac{\partial\Phi(\zeta;q)}{\partial\zeta}\right|_{\zeta=+\infty} = 0 \tag{38}$$

and

$$(1-q)\left.\left[\lambda_0\frac{\partial\Phi(\zeta;q)}{\partial\zeta} - 1\right]\right|_{\zeta=0} = \hbar q \left.\left[\Lambda(q)\frac{\partial\Phi(\zeta;q)}{\partial\zeta} - 1\right]\right|_{\zeta=0}. \tag{39}$$

When $q = 0$, the zero-order deformation equations have the solution

$$\Phi(\zeta;0) = s_0(\zeta), \quad \Lambda(0) = \lambda_0. \tag{40}$$

When $q = 1$, the zero-deformation equations (37)–(39) are the same as the original ones (29) and (30), provided

$$\Phi(\zeta;1) = s(\zeta), \quad \Lambda(1) = \lambda. \tag{41}$$

Therefore, as q increases from 0 to 1, $\Phi(\zeta;q)$ varies or deforms from the initial guess $s_0(\zeta)$ to the exact solution $s(\zeta)$ governed by (29) and (30), so does $\Lambda(q)$ from λ_0 to λ, respectively. This is the basic idea of the homotopy and these kinds of variations are called deformations in topology.

Thus, by Taylor's theorem and (40), we have

$$\Phi(\zeta;q) \sim s_0(\zeta) + \sum_{n=1}^{+\infty} s_n(\zeta)q^n, \tag{42}$$

$$\Lambda(q) \sim \lambda_0 + \sum_{n=1}^{+\infty} \lambda_n q^n, \tag{43}$$

where

$$s_n(\zeta) = \frac{1}{n!}\left.\frac{\partial^n\Phi(\zeta;q)}{\partial q^n}\right|_{q=0}, \quad \lambda_n = \frac{1}{n!}\left.\frac{\partial^n\Lambda(q)}{\partial q^n}\right|_{q=0}. \tag{44}$$

We emphasize that the zero-order deformation equations contain an auxiliary parameter $\hbar$, whose value we have great freedom to choose. Assume that $\hbar$ is so properly chosen that above two series are convergent at $q = 1$, then due to (41) we have (and can prove) that

$$s(\zeta) = s_0(\zeta) + \sum_{n=1}^{+\infty} s_n(\zeta), \tag{45}$$

$$\lambda = \lambda_0 + \sum_{n=1}^{+\infty} \lambda_n. \tag{46}$$

Differentiating the zero-order deformation equations (37)–(39) m times with respect to ζ and then dividing them by $m!$ and finally setting $q = 0$, we have the mth-order deformation equation

$$\mathscr{L}[s_m(\zeta) - \chi_m s_{m-1}(\zeta)] = \hbar R_m(\zeta, \lambda_0, \lambda_1, \ldots, \lambda_{m-1}), \tag{47}$$

subject to the boundary conditions

$$s_m(0) = 0, \quad s'_m(+\infty) = 0 \tag{48}$$

and

$$s'_m(0) = \chi_m s'_{m-1}(0) + \left(\frac{\hbar}{\lambda_0}\right)\left[\sum_{n=0}^{m-1} \lambda_n s'_{m-1-n}(0) - (1-\chi_m)\right] \tag{49}$$

under the definitions

$$R_m = \sum_{n=0}^{m-1} \left[\lambda_n s'''_{m-1-n}(\zeta) + s_n(\zeta)s''_{m-1-n}(\zeta) - \beta s'_n(\zeta)s'_{m-1-n}(\zeta)\right] \tag{50}$$

and

$$\chi_k = \begin{cases} 0, & k \leqslant 1, \\ 1, & k > 1. \end{cases} \tag{51}$$

It is easy to solve the linear mth-order deformation equations (47) and (48). We emphasize that, if the term R_m of (47) contains the term $\exp(-\zeta)$, then, due to the property (35), the solution $s_m(\zeta)$ has the term

$$\zeta \exp(-\zeta),$$

which however *disobeys* the *Rule of Solution Expression A* described by (32), although it is not the so-called secular term in the traditional meaning. So, under the *Rule of Solution Expression A*, we had to force the coefficient of the term $\exp(-\zeta)$ in R_m to be zero. This just provides us with an algebraic equation for λ_{m-1}. Then, one can further get $s_m(\zeta)$. In this way we obtain $\lambda_0, s_1(\zeta), \lambda_1, s_2(\zeta), \ldots,$ successively.

It is found that $s_m(\zeta)$ can be expressed by

$$s_m(\zeta) = \sum_{n=0}^{m+1} c_{m,n} \exp(-n\zeta), \tag{52}$$

where $c_{m,n}$ is coefficient. Due to (33), we have the first two coefficients

$$c_{0,0} = 1/\lambda_0, \quad c_{0,1} = -1/\lambda_0. \tag{53}$$

Substituting the expression (52) into (50), we have

$$R_m = (D_{m,1} - C_{m,1})\exp(-\zeta) - \sum_{n=2}^{m+1} \left(\chi_{m+2-n}C_{m,n} - D_{m,n} + \beta E_{m,n}\right)\exp(-n\zeta), \tag{54}$$

where

$$C_{m,n} = \sum_{k=0}^{m-n} \lambda_k n^3 c_{m-1-k,n}, \tag{55}$$

$$D_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-m+k\}}^{\min\{k+1,n-1\}} (n-j)^2 c_{k,j} c_{m-1-k,n-j}, \tag{56}$$

$$E_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{1,n-m+k\}}^{\min\{k+1,n-1\}} j(n-j) c_{k,j} c_{m-1-k,n-j}. \tag{57}$$

Under the *Rule of Solution Expression A* described by (32), it must hold

$$D_{m,1} - C_{m,1} = 0, \tag{58}$$

which provides us with an algebraic equation for λ_{m-1}. When $m = 1$, this equation gives

$$\lambda_0 = 1. \tag{59}$$

When $m \geqslant 2$, we have, due to (58) and the definitions (55) and (53), the recurrence formula

$$\lambda_{m-1} = \lambda_0\left(\sum_{n=0}^{m-2} \lambda_n c_{m-1-n,1} - D_{m,1}\right). \tag{60}$$

Then, R_m becomes

$$R_m = -\sum_{n=2}^{m+1} \left(\chi_{m+2-n}C_{m,n} - D_{m,n} + \beta E_{m,n}\right)\exp(-n\zeta). \tag{61}$$

Solving the mth-order deformation equations (47)–(49) under above expression of R_m, we have the recurrence formulas

$$c_{m,n} = \chi_m \chi_{m+2-n} c_{m-1,n} + \frac{\hbar\left(\chi_{m+2-n}C_{m,n} - D_{m,n} + \beta E_{m,n}\right)}{\lambda_0 n(n^2-1)} \tag{62}$$

for $2 \leqslant n \leqslant m+1$, and

$$c_{m,1} = -s'_m(0) - \sum_{n=2}^{m+1} n c_{m,n}, \tag{63}$$

$$c_{m,0} = -\sum_{n=1}^{m+1} c_{m,n}, \tag{64}$$

where

$$s'_m(0) = -\chi_m \sum_{k=1}^{m} k c_{m-1,k} - \left(\frac{\hbar}{\lambda_0}\right) \times \left[(1-\chi_m) + \sum_{n=0}^{m-1}\sum_{k=1}^{m-n} k\lambda_n c_{m-1-n,k}\right]. \tag{65}$$

Therefore, we obtain such an *explicit* analytic solution of (14) and (15), i.e.

$$F(\xi) = \sum_{m=0}^{+\infty}\sum_{n=0}^{m+1} c_{m,n} \exp(-n\lambda\xi), \tag{66}$$

where λ is given by (46) under the definition (60), and the coefficients $c_{m,n}$ are calculated first by (53) and then by the recurrence formulas (62)–(64) under the definitions (65) and (55)–(57). At the Mth-order of approximation, we have

$$F(\xi) \approx \sum_{m=0}^{M} \sum_{n=0}^{m+1} c_{m,n} \exp(-n\lambda\xi), \tag{67}$$

$$F'(\xi) \approx -\lambda \sum_{m=0}^{M} \sum_{n=1}^{m+1} n c_{m,n} \exp(-n\lambda\xi) \tag{68}$$

and

$$F''(0) \approx \lambda^2 \sum_{m=0}^{M} \sum_{n=1}^{m+1} n^2 c_{m,n}, \tag{69}$$

where

$$\lambda \approx \sum_{m=0}^{M-1} \lambda_n. \tag{70}$$

Note that $F''(0)$ has clear physical meanings. For a porous medium the value of $F''(0)$ represents the reduced wall heat flux (because $F''(0) = \tau'(0)$) while for the stretched wall it represents the skin friction at the wall.

So far we talk rather little about the auxiliary parameter $\hbar$, which in fact plays a very important role in the frame of the homotopy analysis method. Obviously, it is very important to ensure that the explicit analytic solution (66) is valid for all $\xi \geqslant 0$ in a large enough region of β. In fact, it is the auxiliary parameter $\hbar$ which provides us with a simple way to *control* the convergence of approximation series and to *adjust* convergence regions when necessary.

Liao [19] proved in rather general cases that convergent series given by the homotopy analysis method (at $q = 1$) must be one of exact solutions of a considered non-linear problem. And it is easy to check whether or not a series is convergent in a given region. So, it is convenient to investigate the relationship between the convergence of the explicit analytic solution (66) and the value of the auxiliary parameter $\hbar$, especially by means of symbolic computation software such as mathematica, Maple and so on.

Physically speaking, $F''(0)$ and $F'(\xi)$ are important quantities. Note that $F''(0)$ is a function of β. Our calculations indicate that solution (66) is divergent when $\hbar > 0$. When $\hbar = -1$, corresponding to the traditional way to construct a homotopy, $F''(0)$ given by (69) is convergent in the region $-2 < \beta < 1$, as shown in Fig. 1. However, when $\hbar = -1/2$, the convergence region of $F''(0)$ is enlarged to $-2 < \beta \leqslant 3$, and when $\hbar = -1/5$, the convergence region is further enlarged to $-2 < \beta \leqslant 9$, as shown in Fig. 1. Our calculations indicate that, the closer $\hbar$ is to 0 from below, the larger the convergence region of $F''(0)$ becomes, but the higher-order of approximations is needed to get accurate enough approximations. It seems that the convergence region of $F''(0)$ would tend to infinity as $\hbar \to 0$ from below, although it is not an efficient expression for large β. This clearly verifies that the auxiliary parameter $\hbar$ indeed provides us with a simple way to *control* convergence of approximation series and to *adjust* convergence regions when necessary.

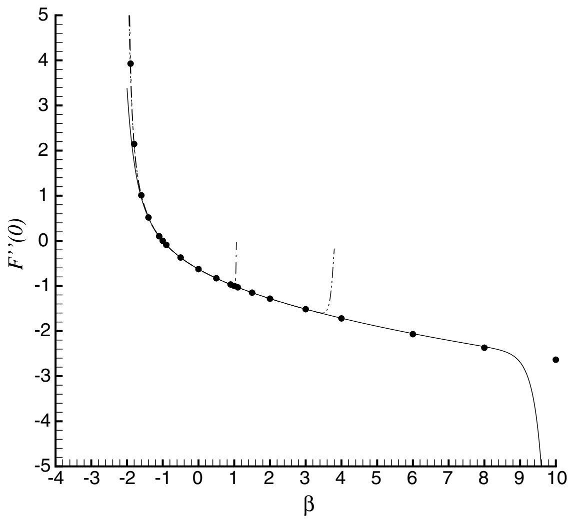

Fig. 1. Comparison of the numerical results of $F''(0)$ with the 50th-order of approximation (69). Symbols: numerical results given by Banks [7]; dash-dotted line: analytic approximation when $\hbar = -1$; dash-dot-dotted line: analytic approximation when $\hbar = -1/2$; solid line: analytic approximation when $\hbar = -1/5$.

For the viscous flow in a porous medium, the solution has physical meaning only when $-2 < \beta \leqslant 2$. So, choosing $\hbar = -1/2$, the explicit analytic solution (66) is convergent to the corresponding numerical results in the whole region of the physical parameter $-2 < \beta \leqslant 2$, as shown in Figs. 1 and 2, so that it can be regarded as an exact analytic solution of Cheng and Minkowyez's equation [15] for viscous flow in a porous medium. Note that, as the non-linearity of the considered problem becomes stronger, higher-order approximations are necessary to give accurate enough approximations, as shown in Fig. 2.

Ingham and Brown [16] numerically found dual solutions of Cheng and Minkowyez's equation [15] for $\beta > 0$. Our analytic solution (66) agrees well with one branch of their numerical results with the property $F'(\xi) \geqslant 0$ for all $\xi \geqslant 0$. However, our analytic approach mentioned above can not give another branch of solutions, which contain negative values of the velocity $F'(\xi)$ or temperature θ in some regions and this is not physically realistic, as explained later.

To investigate the multiple solutions of equations (14) and (15) when $\beta > 0$, more degrees of freedom should be introduced in the frame of the homotopy analysis method. Besides, our analytic solution (66) is not an efficient expression for large β. So, it is worthwhile giving another kind of analytic solutions of (14)

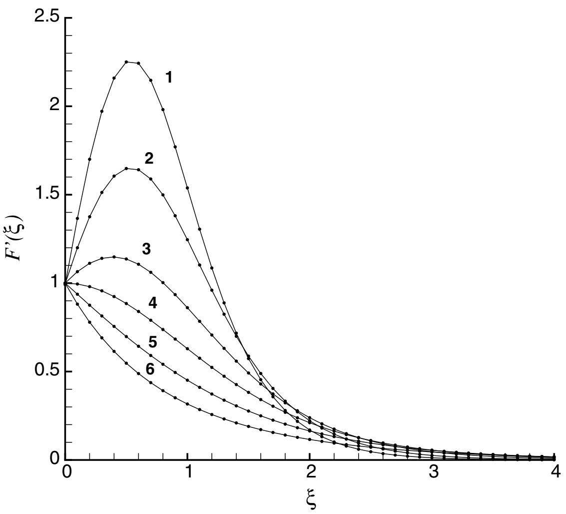

Fig. 2. Comparison of the numerical results of $F'(\xi)$ with the analytic approximations (68) when $\hbar = -1/2$. Symbols: numerical results; curve 1: 70th-order analytic approximation when $\beta = -1.9$; curve 2: 50th-order analytic approximation when $\beta = -1.8$; curve 3: 30th-order analytic approximation when $\beta = -1.5$; curve 4: 15th-order analytic approximation when $\beta = -1$; curve 5: 5th-order analytic approximation when $\beta = 0$; curve 6: 5th-order analytic approximation when $\beta = 2$.

and (15) for all $\beta \geqslant 0$, which has physical meanings for the viscous flows for a stretching wall described in the 2nd section.

3.2. Boundary-layer flows above a stretching wall

Let γ denote a positive constant. Due to the boundary conditions (15), $F(\xi)$ can be expressed by a set of base functions

$$\{\exp(-n\gamma\xi) | n \geqslant 0\} \tag{71}$$

in the following form

$$F(\xi) = a_0 + \sum_{n=1}^{+\infty} a_n \exp(-n\gamma\xi), \tag{72}$$

where a_n is coefficient. This provides us with the *Rule of Solution Expression B*.

Under the *Rule of Solution Expression B* and due to the boundary conditions (15), it is straightforward to choose

$$F_0(\xi) = [1 - \exp(-\gamma\xi)]/\gamma \tag{73}$$

as the initial guess of $F(\xi)$. Besides, due to *Rule of Solution Expression B* described by (72) and the governing equation (14), we choose

$$\widetilde{\mathscr{L}}[\widetilde{\Phi}(\xi;q)] = \exp(\gamma\xi)\left[\frac{\partial^3\widetilde{\Phi}(\xi;q)}{\partial\xi^3} - \gamma^2\frac{\partial\widetilde{\Phi}(\xi;q)}{\partial\xi}\right], \tag{74}$$

as our auxiliary linear operator, where q is an embedding parameter. Note that the auxiliary linear operator $\widetilde{\mathscr{L}}$ has the property

$$\widetilde{\mathscr{L}}[C_1 + C_2\exp(-\gamma\xi) + C_3\exp(\gamma\xi)] = 0. \tag{75}$$

Furthermore, due to (14), we define such a non-linear operator

$$\widetilde{\mathscr{N}}\left[\widetilde{\Phi}(\xi;q)\right] = \frac{\partial^3\widetilde{\Phi}(\xi;q)}{\partial\xi^3} + \widetilde{\Phi}(\xi;q)\frac{\partial^2\widetilde{\Phi}(\xi;q)}{\partial\xi^2} - \beta\left[\frac{\partial\widetilde{\Phi}(\xi;q)}{\partial\xi}\right]^2. \tag{76}$$

Then, we construct the zero-order deformation equations

$$(1-q)\widetilde{\mathscr{L}}\left[\widetilde{\Phi}(\xi;q) - F_0(\xi)\right] = \hbar q\widetilde{\mathscr{N}}\left[\widetilde{\Phi}(\xi;q)\right], \quad q \in [0,1], \tag{77}$$

subject to the boundary conditions

$$\widetilde{\Phi}(0;q) = 0, \quad \left.\frac{\partial\widetilde{\Phi}(\xi;q)}{\partial\xi}\right|_{\xi=0} = 1, \quad \left.\frac{\partial\widetilde{\Phi}(\xi;q)}{\partial\xi}\right|_{\xi=+\infty} = 0. \tag{78}$$

Similarly, we have the relationship

$$F(\xi) = F_0(\xi) + \sum_{m=1}^{+\infty} F_m(\xi), \tag{79}$$

where

$$F_m(\xi) = \frac{1}{m!}\left.\frac{\partial^m\widetilde{\Phi}(\xi;q)}{\partial q^m}\right|_{q=0} \tag{80}$$

is governed by the corresponding mth-order deformation equation

$$\widetilde{\mathscr{L}}[F_m(\xi) - \chi_m F_{m-1}(\xi)] = \hbar\widetilde{R}_m(\xi), \tag{81}$$

subject to the boundary conditions

$$F_m(0) = 0, \quad F'_m(0) = 0, \quad F'_m(+\infty) = 0 \tag{82}$$

under the definitions

$$\widetilde{R}_m(\xi) = F'''_{m-1}(\xi) + \sum_{n=0}^{m-1}\left[F_n(\xi)F''_{m-1-n}(\xi) - \beta F'_n(\xi)F'_{m-1-n}(\xi)\right]. \tag{83}$$

Under the *Rule of Solution Expression B* described by (72), the term $\widetilde{R}_m(\xi)$ contains $\exp(-\gamma\xi)$. However, due to the definition (74), Eq. (81) becomes

$$\left[\frac{\partial^3}{\partial\xi^3} - \gamma^2\frac{\partial}{\partial\xi}\right][F_m(\xi) - \chi_m F_{m-1}(\xi)] = \hbar\exp(-\gamma\xi)\widetilde{R}_m(\xi), \tag{84}$$

whose righthand side

$$\exp(-\gamma\xi)\widetilde{R}_m(\xi)$$

does not contain the term $\exp(-\gamma\xi)$ at all! In this way, the *Rule of Solution Expression B* described by (72) is obeyed. This indicates the flexibility of the homotopy analysis method.

It is found that $F_m(\xi)$ governed by (81) and (82) can be expressed by

$$F_m(\xi) = \sum_{n=0}^{2m+1} a_{m,n} \exp(-n\gamma\xi), \tag{85}$$

where $a_{m,n}$ is coefficient. Substituting above expression into the mth-order deformations (81) and (82), we have the recurrence formulas

$$a_{m,n} = \chi_m \chi_{2m+1-n} a_{m-1,n} + \frac{\hbar\left[\chi_{2m+2-n}\gamma(n-1)^3 a_{m-1,n-1} - A_{m,n-1} + \chi_{n-1}\beta B_{m,n-1}\right]}{\gamma n(n^2-1)} \tag{86}$$

for $2 \leqslant n \leqslant 2m+1$, and

$$a_{m,1} = -\sum_{n=2}^{2m+1} n a_{m,n}, \tag{87}$$

$$a_{m,0} = -\sum_{n=1}^{2m+1} a_{m,n} \tag{88}$$

with the definitions

$$A_{m,j} = \sum_{n=0}^{m-1} \sum_{i=\max\{0,j-2m+2n+1\}}^{\min\{2n+1,j-1\}} (j-i)^2 a_{n,i} a_{m-n-1,j-i}, \quad 1 \leqslant j \leqslant 2m \tag{89}$$

and

$$B_{m,j} = \sum_{n=0}^{m-1} \sum_{i=\max\{1,j-2m+2n+1\}}^{\min\{2n+1,j-1\}} i(j-i) a_{n,i} a_{m-n-1,j-i}, \quad 2 \leqslant j \leqslant 2m. \tag{90}$$

Due to the initial guess (73), we have the first two coefficients

$$a_{0,0} = 1/\gamma, \quad a_{0,1} = -1/\gamma. \tag{91}$$

Thus, starting from these two coefficients, we can calculate the coefficients $a_{m,n}$ for $m = 1, 2, 3, \ldots, 0 \leqslant n \leqslant 2m+1$ by means of the above recurrence formulas. So, we have the second kind of *explicit* analytic solution

$$F(\xi) = \sum_{k=0}^{+\infty} \sum_{n=0}^{2k+1} a_{k,n} \exp(-n\gamma\xi). \tag{92}$$

At the mth-order of approximation we have

$$F(\xi) \approx \sum_{k=0}^{m} \sum_{n=0}^{2k+1} a_{k,n} \exp(-n\gamma\xi), \tag{93}$$

which gives

$$F'(\xi) \approx -\gamma \sum_{k=0}^{m} \sum_{n=1}^{2k+1} n a_{k,n} \exp(-n\gamma\xi) \tag{94}$$

and

$$F''(0) \approx \gamma^2 \sum_{k=0}^{m} \sum_{n=1}^{2k+1} n^2 a_{k,n}. \tag{95}$$

Unlike (66), the explicit analytic solution (92) has two parameters to be chosen. One is the auxiliary parameter $\hbar$. The other is the parameter γ. It is natural to choose $\gamma = 1$ at first. When $\gamma = 1$ and $\hbar$ is a negative constant, we find that the convergence region of the analytic approximation $F''(0)$ given by (95) is strongly dependent upon the value of $\hbar$ and becomes larger as $\hbar$ tends to zero from below, as shown in Fig. 3. When $\gamma = 1$ and $\hbar$ is a function of β such as $\hbar = -1/(1+\beta/7)$, the higher the order of approximation, the larger the convergence regions, as shown in Fig. 4, indicating that the convergence region about β might become infinity as the order of approximation tends to infinity. Thus, when $\gamma = 1$ and $\hbar = -1/(1+\beta/7)$, the explicit analytic solution (92) is valid for $0 \leqslant \beta < +\infty$, as shown in Fig. 5 for $F'(\xi)$.

Note that we have freedom to choose other values of γ. Due to the definition (73) of $F_0(\xi)$, we have

$$F_0(+\infty) = 1/\gamma.$$

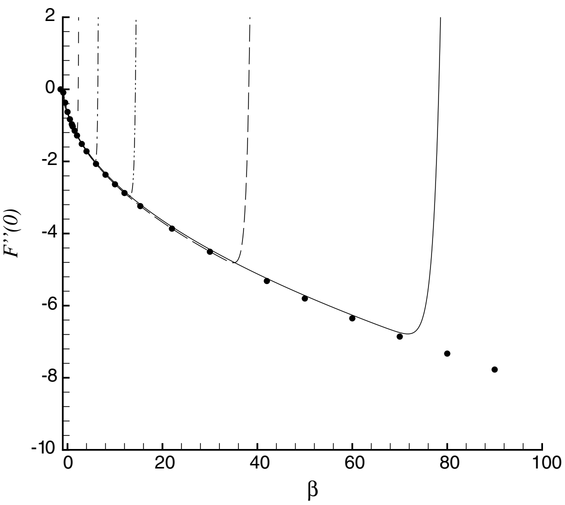

Fig. 3. Comparison of numerical results of $F''(0)$ with 30th-order analytic approximation (95) when $\gamma = 1$. Symbols: numerical results; dashed line: $\hbar = -2$; dash-dotted line: $\hbar = -1$; dash-dot-dotted line: $\hbar = -1/2$; long-dashed line: $\hbar = -1/5$; solid line: $\hbar = -1/10$.

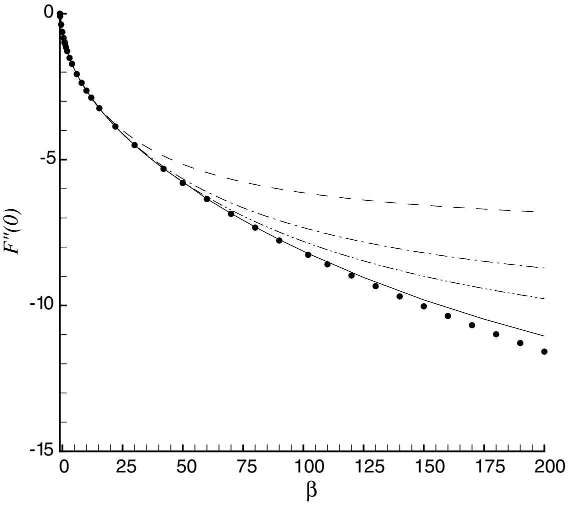

Fig. 4. Comparison of the numerical results of $F''(0)$ with the analytic approximation (95) when $\gamma = 1$ and $\hbar = -(1+\beta/7)^{-1}$. Symbols: numerical results; dashed line: 10th-order approximation; dash-dotted line: 20th-order approximation; dash-dot-dotted line: 30th-order approximation; solid line: 60th-order approximation.

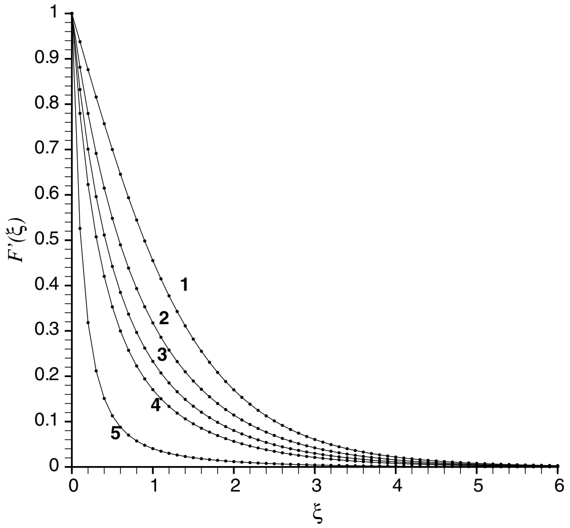

Fig. 5. Comparison of numerical results of $F'(\xi)$ with the analytic approximation (94) when $\gamma = 1$ and $\hbar = -(1+\beta/7)^{-1}$. Symbols: numerical results; curve 1: 20th-order approximation when $\beta = 0$; curve 2: 5th-order approximation when $\beta = 2$; curve 3: 10th-order approximation when $\beta = 5$; curve 4: 10th-order approximation when $\beta = 10$; curve 5: 20th-order approximation when $\beta = 100$.

On the other side, the exact solutions (26) and (27) provide us with the exact value $F(+\infty) = 1$ when $\beta = 1$ and $F(+\infty) = \sqrt{2}$ when $\beta = -1$. So, we choose

$$\gamma = \frac{\sqrt{\beta+3}}{2} \tag{96}$$

to enforce

$$F_0(+\infty) = \frac{2}{\sqrt{\beta+3}}$$

to be above-mentioned values of the exact solutions given by Crane [6]. It is a surprise for us to find that, when $\gamma = \sqrt{\beta+3}/2$ and $\hbar = -1$, even the 3rd-order approximation

$$F''(0) = -\frac{145293 + 231153\beta + 94999\beta^2 + 12395\beta^3}{15120(\beta+3)^{5/2}} \tag{97}$$

agrees quite well with the numerical results for $0 \leqslant \beta \leqslant 1000$, as shown in Fig. 6. The agreement of above expression with numerical results is so good in such a large region that it could be valid in the whole region $0 \leqslant \beta < +\infty$. This verifies once again the validity of the homotopy analysis method.

However, so far we just find the same branch of the solutions as the numerical ones reported by Banks [7], with the property $F'(\xi) \geqslant 0$ for $\xi \geqslant 0$. We attempt a lots of different values of γ and $\hbar$ but found that, as long as an approximation series is convergent, it must converge to the branch of solutions with the property $F'(\xi) \geqslant 0$ for $\xi \geqslant 0$. Besides, we even attempt a more general initial guess

$$F_0(\xi) = [1 - \exp(-\gamma\xi)]/\gamma + \delta[1 - \exp(-\gamma\xi)]^2, \tag{98}$$

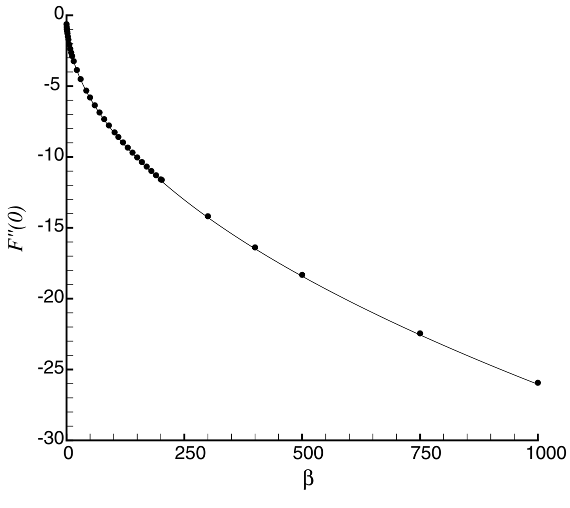

Fig. 6. Comparison of numerical results of $F''(0)$ with the 3rd-order analytic approximation (97) when $\gamma = \sqrt{3+\beta}/2$ and $\hbar = -1$. Symbols: numerical results; solid line: 3rd-order analytic approximation (97).

but still fail to get a result convergent to the 2nd branch of solutions reported by Ingham and Brown [16] with the negative $F'(\xi)$ in some region. Note that both of the explicit analytic solutions (66) and (92) give the same branch of solutions with the property $F'(\xi) \geqslant 0$ for $\xi \geqslant 0$. This might imply that the 2nd branch of numerical solutions reported by Ingham and Brown [16] would not *exponentially* tend to a constant as $\xi \to +\infty$, because, seriously speaking, it is impossible to verify this kind of property at *infinity* by *numerical* methods.

Note that, for a porous medium, it holds

$$T = T_\infty + x^\kappa (T_w - T_\infty)\theta.$$

So, $T < T_\infty$ in some region for negative temperature θ when $T_w > T_\infty$. Assume that the fluid is water and the temperature T_∞ is just 0.0001 °C and the wall is suddenly heated to the temperature T_w. The first branch of solutions indicates that the temperature decays monotonously from T_w to T_∞, which is physically realistic. However, the second branch of solutions indicates that the temperature near the wall, although the wall is suddenly *heated* to T_w, is below zero so that the water at that region is *frozen* and the solution becomes invalid. So, the second branch of solutions seems not physically realistic.

4. Discussion and conclusions

In this paper the homotopy analysis method is successfully applied to give two kinds of explicit analytic solutions of the boundary-layer equations, which is valid not only for the convective viscous flow past a suddenly heated vertical plate in a porous medium but also for the viscous flow over a stretching wall. The solution (66) when $\hbar = -1/2$ is valid for $-2 < \beta \leqslant 2$ that covers the *whole* region of β having physical meanings for the viscous flows in a porous medium. The solution (92) when $\hbar = -1/(1+\beta/7)$ and $\gamma = 1$ is valid for $0 \leqslant \beta < +\infty$, and even the 3rd-order approximation (97) of $F''(0)$ when $\gamma = \sqrt{\beta+3}/2$ and $\hbar = -1$ gives accurate results for $0 \leqslant \beta < +\infty$. Note that the two kinds of analytic solutions are *explicitly* given by recurrence formulas. To the best of our knowledge, it is the first time that such kinds of explicit analytic solutions of Eqs. (14) and (15) are reported. Because many phenomenon can be described by Eqs. (14) and (15), our analytic solutions (66) and (92) may find their wide applications in science and engineering.

Our approach gives the solutions with the property $F'(\xi) \geqslant 0$ for $\xi \geqslant 0$. However, both of our two kinds of solutions can not give the second branch of numerical solutions with the property $F'(\xi) < 0$ in some region of ξ, reported by Ingham and Brown [16]. It seems possible that the second branch of solution does not decay *exponentially*. It seems also possible that the second branch of solutions is not physically realistic. Thus, it is worthwhile further investigating the multiple solutions of Eqs. (14) and (15) and giving the physical meanings of the second branch of solutions reported by Ingham and Brown [16].

Note that our analytic solutions contain the auxiliary parameter $\hbar$. It is the auxiliary parameter $\hbar$ which provides us with a simple way to *control* convergence of approximation series and to *adjust* convergence regions, as shown in this paper. This is the advantage of the homotopy analysis method over *all* other perturbation and non-perturbation methods. The success of the homotopy analysis method for considered problems verifies once again that it is indeed a useful analytic tool for non-linear problems in science and engineering, although further improvements are necessary.

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